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Conditions for linearization of a projectable system of two second-order ordinary differential equations

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Abstract

A new method for linearizing a system of ordinary differential equations is introduced. The method is applied to a system of two second-order ordinary differential equations. It is shown that for a particular class of equations, the method gives more general linearization criteria than linearization via point transformations. Examples of systems of equations which are not linearizable via point transformations, but linearizable by the new method, are given.

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1. Introduction

1.1. One second-order ODE

In mathematical history [1], Lie was the first to study the linearization problem of second-order ordinary differential equations (ODEs). He gave the linearization criteria for a second-order ordinary differential equation to be transformed into the simplest linear equation ($\ddot{u} = 0$) by an invertible point transformation of the independent and dependent variables,

$$t = \varphi(x, y), \qquad u = \psi(x, y). \tag{1}$$

He showed that every linearizable second-order ordinary differential equation has the form

$$y'' = a(x, y)y'^3 + b(x, y)y'^2 + c(x, y)y' + d(x, y), \tag{2}$$

where $y' = \frac{dy}{dx}$, $y'' = \frac{d^2y}{dx^2}$ and the coefficients $a(x, y)$, $b(x, y)$, $c(x, y)$ and $d(x, y)$ satisfy the conditions $H = 0$ and $K = 0$, where

$$\begin{aligned} H &= 2b_{xy} - 3a_{xx} - c_{yy} - 3a_x c + 3a_y d + 2b_x b - 3c_x a - c_y b + 6d_y a, \\ K &= 2c_{xy} - b_{xx} - 3d_{yy} - 6a_x d + b_x c + 3b_y d - 2c_y c - 3d_x a + 3d_y b. \end{aligned} \tag{3}$$

The functions H and K are relative invariants [2] with respect to invertible transformation (1).

There exist various approaches for solving the linearization problem of a second-order ordinary differential equation. For example, one was developed by Cartan [3], who used differential geometry for solving this problem. Another approach makes use of the generalized Sundman transformation [4]. All the approaches have also been applied to third-order and fourth-order ordinary differential equations [5–12][1].

1.2. System of two second-order ODEs

The linearization problem of a system of two second-order ordinary differential equations

$$\ddot{x} = G(t, x, y, \dot{x}, \dot{y}), \qquad \ddot{y} = F(t, x, y, \dot{x}, \dot{y}) \tag{4}$$

consists of finding an invertible transformation, which transforms system of equations (4) into a linear system of equations. Here $\dot{x} = \frac{dx}{dt}$, $\ddot{x} = \frac{d^2x}{dt^2}$, $\dot{y} = \frac{dy}{dt}$ and $\ddot{y} = \frac{d^2y}{dt^2}$. The linearization problem for system (4), using a point transformation

$$v = \varphi(t, x, y), \qquad u_1 = \psi_1(t, x, y), \qquad u_2 = \psi_2(t, x, y), \tag{5}$$

was studied in [14–16] and [17]. In [14], criteria of linearization of system (4) are related to the existence of admitted four-dimensional Lie algebra. In [15], criteria for the system (4) to be equivalent to the trivial system of two second-order ordinary differential equations ($\ddot{u}_1 = 0, \ddot{u}_2 = 0$) were given. Linearization criteria for a system of two second-order quadratically semi-linear ordinary differential equations were studied in [16]. In [17], it is shown that if system (4) is linearizable, then it must be of the form

$$\begin{aligned} &\ddot{x} + \dot{x}(a_{11}\dot{x}^2 + a_{12}\dot{x}\dot{y} + a_{13}\dot{y}^2) + a_{14}\dot{x}^2 + a_{15}\dot{x}\dot{y} + a_{16}\dot{y}^2 + a_{17}\dot{x} + a_{18}\dot{y} + a_{19} = 0,\\ &\ddot{y} + \dot{y}(a_{11}\dot{x}^2 + a_{12}\dot{x}\dot{y} + a_{13}\dot{y}^2) + a_{24}\dot{x}^2 + a_{25}\dot{x}\dot{y} + a_{26}\dot{y}^2 + a_{27}\dot{x} + a_{28}\dot{y} + a_{29} = 0, \end{aligned} \tag{6}$$

where $a_{ij}(t, x, y)$ are some functions. Direct calculations [17] prove that the form (6) is not changed by any invertible point transformations (5). Some relative invariants of (6) were also obtained in [17].

2. Statement of the problem

A novel method for linearization of two second-order ordinary differential equations (4) with two dependent variables x and y and one independent variable t is proposed in this communication.

Assume that $\dot{x} \neq 0$. Then by virtue of the inverse function theorem, one can consider $y = y(x)$. Substituting the derivatives

$$\dot{y} = y'\dot{x}, \qquad \ddot{y} = y''\dot{x}^2 + y'\ddot{x}$$

into the first equation of (4), and using the second equation of (4), one obtains

$$\dot{x}^2 y'' + y'\tilde{G} - \tilde{F} = 0,$$

where

$$\tilde{G}(t, x, y, \dot{x}, y') = G(t, x, y, \dot{x}, \dot{x}y'), \qquad \tilde{F}(t, x, y, \dot{x}, y') = F(t, x, y, \dot{x}, \dot{x}y'),$$

$$y' = \frac{dy}{dx}, \qquad y'' = \frac{d^2y}{dx^2}.$$

Suppose that

$$\tilde{F}(t, x, y, \dot{x}, y') - y'\tilde{G}(t, x, y, \dot{x}, y') = \dot{x}^2\lambda(x, y, y'), \tag{7}$$

[1] Review of solving linearization problems can be found in [13].

where $x, y, \dot{x}$ and y' are considered as the independent variables of the functions $\tilde{G}$, $\tilde{F}$ and λ. We call a system (4) satisfying condition (7) a projectable system. This definition of a projectable system of equations can be extended to any normal system of ordinary differential equations. Another extension of the definition can be given as follows. A system of equations (4) is called projectable if there exists an invertible change of the independent and dependent variables $\bar{x} = g_1(t, x, y)$, $\bar{y} = g_2(t, x, y)$ and $\bar{t} = g_3(t, x, y)$ such that the equivalent system possesses property (7). In the present communication, we consider the simple case of a projectable system, where $g_1 = x$, $g_2 = y$ and $g_3 = t$.

Equation (7) requires that the function λ defined by the formula

$$\lambda(x, y, z) = \frac{1}{\dot{x}^2}(F(t, x, y, \dot{x}, z\dot{x}) - zG(t, x, y, \dot{x}, z\dot{x})) \tag{8}$$

only depends on x, y and $z = \frac{\dot{y}}{\dot{x}}$. The function $y(x)$ satisfies the second-order ordinary differential equation

$$y'' = \lambda(x, y, y'). \tag{9}$$

A solution of a projectable system (4) can be found in two sequential steps: in the first step, one solves equation (9); in the second step, one finds a solution $x(t)$ of the first equation of (4) with substituted $y = y(x)$ and $\dot{y} = y'(x)\dot{x}$:

$$\ddot{x} = G(t, x, y(x), \dot{x}, \dot{x}y'(x)). \tag{10}$$

If at each step one has a linearizable second-order ordinary differential equation, then we call system (4) a sequentially linearizable system of equations. In this communication, we give necessary and sufficient conditions for system (4) to be sequentially linearizable.

3. Sequentially linearizable system (4)

Since at each step the equations are second-order ordinary differential equations, one can sequentially apply the Lie criteria to equations (9) and (10).

Theorem. *A projectable system (4) is sequentially linearizable if and only if the functions $\lambda(x, y, z)$ and $G(t, x, y, \dot{x}, \dot{y})$ have the representations*

$$\lambda(x, y, z) = b_1(x, y)z^3 + b_2(x, y)z^2 + b_3(x, y)z + b_4(x, y), \tag{11}$$

$$G(t, x, y, \dot{x}, \dot{y}) = a_1(t, x, y, z)\dot{x}^3 + a_2(t, x, y, z)\dot{x}^2 + a_3(t, x, y, z)\dot{x} + a_4(t, x, y, z), \tag{12}$$

where the coefficients $b_i(x, y)$ and $a_i(t, x, y, z)(i = 1, 2, 3, 4)$ satisfy the equations

$$2b_{2xy} - 3b_{1xx} - b_{3yy} - 3b_{1x}b_3 + 3b_{1y}b_4 + 2b_{2x}b_2 - 3b_{3x}b_1 - b_{3y}b_2 + 6b_{4y}b_1 = 0, \tag{13}$$

$$2b_{3xy} - b_{2xx} - 3b_{4yy} - 6b_{1x}b_4 + b_{2x}b_3 + 3b_{2y}b_4 - 2b_{3y}b_3 - 3b_{4x}b_1 + 3b_{4y}b_2 = 0, \tag{14}$$

$$\sum_{i=1}^{6} \beta_i(z)^{7-i} + \beta_7 = 0, \qquad \sum_{i=1}^{6} \beta_{i+7}(z)^{7-i} + \beta_{14} = 0. \tag{15}$$

Here

$\beta_1 = -a_{3zz}b_1{}^2$, $\beta_2 = b_1(-2a_{3zz}b_2 - 3a_{3z}b_1)$,
$\beta_3 = -2a_{3yz}b_1 - 2a_{3zz}b_1b_3 - a_{3zz}b_2{}^2 - a_{3z}b_{1y} - 5a_{3z}b_1b_2$,
$\beta_4 = 3a_{1z}a_4b_1 + 2a_{2tz}b_1 - 2a_{3xz}b_1 - 2a_{3yz}b_2 - a_{3y}b_1 - 2a_{3zz}b_1b_4 - 2a_{3zz}b_2b_3 - a_{3z}b_{1x} - a_{3z}b_{2y} - a_{3z}a_2b_1 - 4a_{3z}b_1b_3 - 2a_{3z}b_2{}^2 + 6a_{4z}a_1b_1$,
$\beta_5 = 3a_{1z}a_4b_2 + 2a_{2tz}b_2 - 2a_{3xz}b_2 - 2a_{3yz}b_3 - a_{3yy} - a_{3y}b_2 - 2a_{3zz}b_2b_4 - a_{3zz}b_3{}^2 - a_{3z}b_{2x} - a_{3z}b_{3y} - a_{3z}a_2b_2 - 3a_{3z}b_1b_4 - 3a_{3z}b_2b_3 + 6a_{4z}a_1b_2$,
$\beta_6 = 3a_{1y}a_4 + 3a_{1z}a_4b_3 + 2a_{2ty} + 2a_{2tz}b_3 - 2a_{3xy} - 2a_{3xz}b_3 - 2a_{3yz}b_4 - a_{3y}a_2 - a_{3y}b_3 - 2a_{3zz}b_3b_4 - a_{3z}b_{3x} - a_{3z}b_{4y} - a_{3z}a_2b_3 - 2a_{3z}b_2b_4 - a_{3z}b_3{}^2 + 6a_{4y}a_1 + 6a_{4z}a_1b_3$,
$\beta_7 = 3a_{1x}a_4 - 3a_{1tt} - 3a_{1t}a_3 + 3a_{1z}a_4b_4 + 2a_{2xt} + 2a_{2tz}b_4 + 2a_{2t}a_2 - 2a_{3xz}b_4 - a_{3xx} - a_{3x}a_2 - 3a_{3t}a_1 - a_{3y}b_4 - a_{3zz}b_4{}^2 - a_{3z}b_{4x} - a_{3z}a_2b_4 - a_{3z}b_3b_4 + 6a_{4x}a_1 + 6a_{4z}a_1b_4$,
$\beta_8 = -3a_{4zz}b_1{}^2$, $\beta_9 = 3b_1(-2a_{4zz}b_2 - 3a_{4z}b_1)$,
$\beta_{10} = 3(-2a_{4yz}b_1 - 2a_{4zz}b_1b_3 - a_{4zz}b_2{}^2 - a_{4z}b_{1y} - 5a_{4z}b_1b_2)$,
$\beta_{11} = 3a_{2z}a_4b_1 + 2a_{3tz}b_1 - 2a_{3z}a_3b_1 - 6a_{4xz}b_1 - 6a_{4yz}b_2 - 3a_{4y}b_1 - 6a_{4zz}b_1b_4 - 6a_{4zz}b_2b_3 - 3a_{4z}b_{1x} - 3a_{4z}b_{2y} + 3a_{4z}a_2b_1 - 12a_{4z}b_1b_3 - 6a_{4z}b_2{}^2$,
$\beta_{12} = 3a_{2z}a_4b_2 + 2a_{3tz}b_2 - 2a_{3z}a_3b_2 - 6a_{4xz}b_2 - 6a_{4yz}b_3 - 3a_{4yy} - 3a_{4y}b_2 - 6a_{4zz}b_2b_4 - 3a_{4zz}b_3{}^2 - 3a_{4z}b_{2x} - 3a_{4z}b_{3y} + 3a_{4z}a_2b_2 - 9a_{4z}b_1b_4 - 9a_{4z}b_2b_3$,
$\beta_{13} = 3a_{2y}a_4 + 3a_{2z}a_4b_3 + 2a_{3ty} + 2a_{3tz}b_3 - 2a_{3y}a_3 - 2a_{3z}a_3b_3 - 6a_{4xy} - 6a_{4xz}b_3 - 6a_{4yz}b_4 + 3a_{4y}a_2 - 3a_{4y}b_3 - 6a_{4zz}b_3b_4 - 3a_{4z}b_{3x} - 3a_{4z}b_{4y} + 3a_{4z}a_2b_3 - 6a_{4z}b_2b_4 - 3a_{4z}b_3{}^2$,
$\beta_{14} = -6a_{1t}a_4 + 3a_{2x}a_4 - a_{2tt} + a_{2t}a_3 + 3a_{2z}a_4b_4 + 2a_{3xt} - 2a_{3x}a_3 + 2a_{3tz}b_4 - 2a_{3z}a_3b_4 - 6a_{4xz}b_4 - 3a_{4xx} + 3a_{4x}a_2 - 3a_{4t}a_1 - 3a_{4y}b_4 - 3a_{4zz}b_4{}^2 - 3a_{4z}b_{4x} + 3a_{4z}a_2b_4 - 3a_{4z}b_3b_4$.

3.1. Proof of the theorem

For a linearizable second-order ordinary differential equation $y'' = \lambda(x, y, y')$, the function $\lambda(x, y, y')$ must have the form (2)

$$\lambda(x, y, y') = b_1(x, y)y'^3 + b_2(x, y)y'^2 + b_3(x, y)y' + b_4(x, y), \tag{16}$$

where the coefficients $b_i(x, y)(i = 1, 2, 3, 4)$ satisfy the conditions

$$2b_{2xy} - 3b_{1xx} - b_{3yy} - 3b_{1x}b_3 + 3b_{1y}b_4 + 2b_{2x}b_2 - 3b_{3x}b_1 - b_{3y}b_2 + 6b_{4y}b_1 = 0, \tag{17}$$

$$2b_{3xy} - b_{2xx} - 3b_{4yy} - 6b_{1x}b_4 + b_{2x}b_3 + 3b_{2y}b_4 - 2b_{3y}b_3 - 3b_{4x}b_1 + 3b_{4y}b_2 = 0. \tag{18}$$

Assuming that a solution $y(x)$ of the equation $y'' = \lambda(x, y, y')$ with the function λ (16) is given, the first equation of (4) becomes

$$\ddot{x} = G(t, x, y(x), \dot{x}, y'(x)\dot{x}). \tag{19}$$

According to the Lie criteria, equation (19) is linearizable if and only if

$$G(t, x, y(x), \dot{x}, y'(x)\dot{x}) = h_1(t, x)\dot{x}^3 + h_2(t, x)\dot{x}^2 + h_3(t, x)\dot{x} + h_4(t, x), \tag{20}$$

where the coefficients $h_i(t,x) = a_i(t,x,y(x),y'(x))(i = 1,2,3,4)$ satisfy the conditions $H = 0$ and $K = 0$, with $a = h_1, b = h_2, c = h_3, d = h_4$. These conditions become

$$\begin{aligned}2D_x a_{2t} - 3a_{1tt} - D_x^2 a_3 - 3a_{1t}a_3 + 3(D_x a_1)a_4 + 2a_{2t}a_2 \\ - 3a_{3t}a_1 - (D_x a_3)a_2 + 6(D_x a_4)a_1 = 0,\end{aligned} \tag{21}$$

$$\begin{aligned}2D_x a_{3t} - a_{2tt} - 3D_x^2 a_4 - 6a_{1t}a_4 + a_{2t}a_3 + 3(D_x a_2)a_4 \\ - 2(D_x a_3)a_3 - 3a_{4t}a_1 + 3(D_x a_4)a_2 = 0.\end{aligned} \tag{22}$$

Here, the operator D_x is the operator of the total derivative with respect to x:

$$D_x = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + y'''\frac{\partial}{\partial y''}.$$

Substituting $y'' = \lambda$ and $y''' = D_x\lambda$ into equations (21) and (22) respectively, one obtains equations (15).

4. Quadratically semi-linear equations

In this section, we show that a system of two second-order quadratically semi-linear ordinary differential equations

$$\begin{aligned}\ddot{x} &= a(x,y)\dot{x}^2 + 2b(x,y)\dot{x}\dot{y} + c(x,y)\dot{y}^2,\\ \ddot{y} &= d(x,y)\dot{x}^2 + 2e(x,y)\dot{x}\dot{y} + f(x,y)\dot{y}^2,\end{aligned} \tag{23}$$

which is linearizable via point transformations (5), is also sequentially linearizable. Linearization criteria for system (23) via point transformations were obtained in [16]. These criteria are

$$S_i = 0 \quad (i = 1,2,3,4), \tag{24}$$

where

$$S_1 = a_y - b_x + be - cd, \qquad S_2 = b_y - c_x + (ac - b^2) + (bf - ce),$$
$$S_3 = d_y - e_x - (ae - bd) - (df - e^2), \qquad S_4 = b_x + f_x - a_y - e_y.$$

Note that system (23) is a projectable system with

$$\lambda(x,y,y') = -cy'^3 + (f - 2b)y'^2 + (2e - a)y' + d. \tag{25}$$

Applying the above proven theorem, one obtains the conditions for system (23) to be sequentially linearizable:

$$\begin{aligned}&3S_{1y} - 3S_{2x} + 2S_{4y} + 3(f - b)S_1 - 3eS_2 - 3cS_3 + (2f - b)S_4 = 0,\\ &3S_{1x} + 3S_{3y} + S_{4x} - 3(e - a)S_1 + 3dS_2 + 3bS_3 - (2e - a)S_4 = 0.\end{aligned} \tag{26}$$

Relations (24) vanish (26). Thus the quadratically semi-linear system (23) is not only linearizable via point transformations, but also sequentially linearizable. Furthermore, equations (26) show that the set of systems (23), which is linearizable via point transformations, is a particular class of equations which can be sequentially linearizable.

5. Examples

In this section we demonstrate examples of systems of two second-order ordinary differential equations which are sequentially linearizable, but not linearizable via point transformations.

Consider a system

$$\ddot{x} = y, \qquad \ddot{y} = \frac{\dot{y}}{\dot{x}} y. \tag{27}$$

As was mentioned in section 1, if a system of two second-order ordinary differential equations is linearizable by point transformations, then it has to be of the form (6). Since system (27) is not of form (6), system (27) is not linearizable by point transformations. Let us show that system (27) is sequentially linearizable.

For system (27), $\lambda = 0$ which implies that $y'' = 0$. The first equation of (27) becomes $\ddot{x} = c_1 x + c_2$, which is a linear second-order equation. Therefore, system (27) is sequentially linearizable. Note that system (27) is a particular case of the system

$$\begin{aligned} \ddot{x} &= f\left(y - \frac{\dot{y}}{\dot{x}}x, \frac{\dot{y}}{\dot{x}}, t\right) + xg\left(y - \frac{\dot{y}}{\dot{x}}x, \frac{\dot{y}}{\dot{x}}, t\right), \\ \ddot{y} &= \frac{\dot{y}}{\dot{x}}\left(f\left(y - \frac{\dot{y}}{\dot{x}}x, \frac{\dot{y}}{\dot{x}}, t\right) + xg\left(y - \frac{\dot{y}}{\dot{x}}x, \frac{\dot{y}}{\dot{x}}, t\right)\right), \end{aligned} \tag{28}$$

which is also sequentially linearizable. Here, the functions f and g are arbitrary.

The example presented shows that a system of two second-order ordinary differential equations, which is not linearizable by point transformations, might be sequentially linearizable.

Another observation is as follows. System (27) is equivalent to the fourth-order ordinary differential equation:

$$x^{(4)} = \frac{x^{(3)}}{\dot{x}} \ddot{x}. \tag{29}$$

Applying the linearization criteria obtained in [12] to equation (29), one obtains that equation (29) is also not linearizable by point transformations.

6. Conclusion

In this paper, a new method for linearizing a system of ordinary differential equations is introduced. This method consists of a sequentially reducing number of the dependent variables and using linearization criteria for the reduced equations. We call systems linearizable by the new procedure as sequentially linearizable. The method is applied to a system of two second-order ordinary differential equations. Moreover, it is shown that for systems of two second-order quadratically semi-linear ordinary differential equations the new method give a more general set of linearizable systems than via point transformations. Finally, an example of equations which are not linearizable by point transformations, but do sequentially linearize by the new method, is given.

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